*International Journal of Theoretical Physics, Vol. 40, No. 9, September 2001 (© 2001)*

# Spacetime as Manifold of Internal Symmetry Orbits in External Symmetries

**Heinrich Saller**[1,2]

*Received March 21, 2001*

Interactions and particles in the standard model are characterized by the action of internal and external symmetry groups. The four symmetry regimes involved are related to each other in the context of induced group representations. In addition to Wigner's induced representations of external Poincaré group operations, parametrized by energy-momenta, and the induced internal hyperisospin representations, parametrized by the standard model Higgs field, the external operations, including the Lorentz group, can also be considered to be induced by the internal operations of the hypercharge–isospin group. In such an interpretation nonlinear spacetime is parametrized by the orbits of the internal action group in the external action group.

## 1. INTRODUCTION

Particle physics as described in the standard model for electroweak and strong interactions is characterized by four symmetry regimes. First one has the external spacetime related transformation groups, the Lorentz group $\mathbf{SO}_0(1, 3)$ with its double cover $\mathbf{SL}(\mathbb{C}^2)$, and the internal compact groups – $\mathbf{U}(1)$ for hypercharge, $\mathbf{SU}(2)$ for isospin, and $\mathbf{SU}(3)$ for color acting on the quantum fields that describe the interaction. From these symmetries for the interaction, one has to distinguish sharply the external and internal symmetries for the asymptotic particle states. A particle is characterized by one translation eigenvalue, its mass, and one space rotation number, its spin for nontrivial mass, and its polarization for the massless case. The rotation invariants are related to external subgroups – spin $\mathbf{SU}(2)$ and polarization $\mathbf{SO}(2)$. With respect to the internal symmetry only an abelian electromagnetic $\mathbf{U}(1)$ remains as symmetry for the particles. Color symmetry is

[1] Max Planck Institut für Physik and Astrophysik, Werner Heisenberg Institut für Physik, München, Germany.

[2] To whom correspondence should be addressed at Max Planck Institut für Physik and Astrophysik, Werner Heisenberg Institut, Fohringer Ring 6, D-80805 München, Germany; e-mail: hns@mppmu.mpg.de.



0020-7748/01/0900-1545$19.50/0 

confined and hypercharge–isospin is spontaneously broken. The groups involved

Four Symmetry Regimes

| | Internal | External |
|---|---|---|
| Interactions | $\mathbf{U}(1) \circ [\mathbf{SU}(2) \times \mathbf{SU}(3)]$ | $\mathbf{SL}(\mathbb{C}^2)$ |
| Particles | $\mathbf{U}(1)$ | $\mathbf{SU}(2)$, $\mathbf{SO}(2)$ |

show subgroup relations in the vertical direction, that is from particles to interactions.[3] It will be argued later that there is also a horizontal subgroup relation involved, that is from internal to external

$$\begin{array}{ccc} & \text{spacetime} & \\ & x_{\succ} & \\ \mathbf{U}(2) & \longrightarrow & \mathbf{GL}(\mathbb{C}^2) \\ \text{Higgs vectors } \Phi \ \uparrow & & \uparrow \ q \text{ energy-momenta} \\ \mathbf{U}(1) & & \mathbf{D}(1) \times \mathbf{U}(1) \circ \mathbf{SU}(2) \\ & & \mathbf{D}(1) \times \mathbf{U}(1) \times \mathbf{SO}(2) \end{array}$$

In this diagram the external interaction group (upper right) is the Lorentz covering group $\mathbf{SL}(\mathbb{C}^2)$, supplemented with a dilatation group $\mathbf{D}(1) = \exp\mathbb{R}$ (causal group) and a phase group $\mathbf{U}(1) = \exp i\mathbb{R}$ (fermion number group), that is the full linear group $\mathbf{GL}(\mathbb{C}^2)$. A corresponding extension with $\mathbf{GL}(\mathbb{C}) = \mathbf{U}(1) \times \mathbf{D}(1)$ is used for the external particle groups (lower right). For the internal interaction symmetry (upper left), the color group $\mathbf{SU}(3)$ is omitted. The three arrows for inclusion relations will be related below to induced representations. They are labeled with manifold parameters, the Higgs parameters $\Phi \in \mathbb{C}^2$ with $\|\Phi\|^2 = M^2 > 0$ for the internal induction from particle symmetry (lower left) to interaction symmetry; the mass shell energy-momenta $q \in \mathbb{R}^4$, with $q^2 = m^2 \geq 0$ for the corresponding external induction; and strictly future spacetime parameters $x_{\succ} \in \mathbb{R}^4$, with $x_{\succ}^2 > 0$ and $x_{\succ 0} > 0$ parametrizing the induction from internal to external interaction symmetries. To motivate and to understand these transmutations from groups to subgroups is the aim of this paper.

[3] H. saller, hep-th/0010057.

## 2. EXTERNAL TRANSMUTATION FROM PARTICLES TO INTERACTIONS

According to Wigner (1939) particles are embedded into an irreducible definite unitary action of the Poincaré group $\mathbb{R}^4 \overleftarrow{\times} \mathbf{SO}_0(1,3)$ as the semidirect product of the orthochronous Lorentz group $\mathbf{SO}_0(1,3)$ with the spacetime translations $\mathbb{R}^4$. The infinite-dimensional Poincaré group representations are induced (Folland, 1995; Mackey, 1968) by finite-dimensional irreducible representations of direct product subgroups where the homogeneous factor comes from energy-momentum fixgroups ("little groups"): The rest system rotations $\mathbf{SO}(3)$ for energy-momenta $q \in \mathbb{R}^4$ with $q^2 > 0$, the noncompact group $\mathbf{SO}_0(1,2)$ for energy-momenta with $q^2 < 0$, and the axial rotations $\mathbf{SO}(2)$ around the momentum direction as "fixgroup in the fixgroup" $\mathbb{R}^2 \overleftarrow{\times} \mathbf{SO}(2) \subset \mathbf{SO}_0(1,3)$ for nontrivial energy-momenta with $q^2 = 0$. Only particles with the representations for causal momenta $q^2 \geq 0$ and compact little groups $\mathbf{SO}(3)$ and $\mathbf{SO}(2)$ are found.

With respect to the halfinteger spin particles the twofold covering simply connected groups $\mathbf{SL}(\mathbb{C}^2) \supset \mathbf{SU}(2)$ for $\mathbf{SO}_0(1,3) \supset \mathbf{SO}(3)$ are used

$$u \in \mathbf{SU}(2) \Rightarrow O(u)^a_b = \frac{1}{2}\text{tr}\, u\sigma^a u^* \sigma^b \in \mathbf{SO}(3) \qquad \cong \mathbf{SU}(2)/\mathbb{I}(2)$$

$$s \in \mathbf{SL}(\mathbb{C}^2) \Rightarrow \Lambda(s)^i_j = \frac{1}{2}\text{tr}\, s\sigma^i s^* \check{\sigma}_j \in \mathbf{SO}_0(1,3) \quad \cong \mathbf{SL}(\mathbb{C}^2)/\mathbb{I}(2)$$

with centr $\mathbf{SL}(\mathbb{C}^2) =$ centr $\mathbf{SU}(2) = \mathbb{I}(2) = \{\pm \mathbf{1}_2\}$
The traces involve the hermitian

$$\text{Pauli–Weyl matrices: } \sigma^i = (\mathbf{1}_2, \sigma^a) = \check{\sigma}_i, \quad \begin{cases} a = 1, 2, 3 \\ i = 0, 1, 2, 3 \end{cases}$$

Therewith the twofold cover of the Poincaré group comes with the multiplication law

$$\mathbb{R}^4 \overleftarrow{\times} \mathbf{SL}(\mathbb{C}^2) \ni (x, s) \quad \text{with} \quad \begin{cases} x = x_j \sigma^j = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \in \mathbb{R}^4 \\ (x, s) \circ (x', s') = (x + s \circ x' \circ s^*, s \circ s') \end{cases}$$

The induction procedure used for massive and massless particles is symbolized with the representation equivalence classes **rep** as follows

$$\mathbf{rep}[\mathbb{R} \times \mathbf{SU}(2)] \uplus \mathbf{rep}[\mathbb{R} \times \mathbf{SO}(2)] \stackrel{\text{ind}}{\longmapsto} \mathbf{rep}[\mathbb{R}^4 \overleftarrow{\times} \mathbf{SL}(\mathbb{C}^2)]$$

The discrete invariants $2J \in \mathbb{N}$ for $\mathbf{SU}(2)$ and $\pm z \in \mathbb{Z}$ for $\mathbf{SO}(2)$ give spin and polarization respectively, the continuous invariant $q^2 = m^2 \geq 0$ for the translations gives the corresponding mass. According to Wigner's particle definition, confined quarks are no particles.

In the case of spin **SU**(2), the transition from a massive particle rest system, defining a time direction, to the Lorentz group action compatible framework is performed with the boost representations, parametrized by the three real numbers in the energy-momenta $\frac{q}{m}$

$$s\left(\frac{q}{m}\right) = e^{\frac{\vec{\beta}\vec{\sigma}}{2}} = \sqrt{\frac{m+q_0}{2m}}\left[\mathbf{1}_2 + \frac{\vec{q}\vec{\sigma}}{m+q_0}\right] \in \mathbf{SL}(\mathbb{C}^2)$$

$$\text{with} \begin{cases} \vec{\beta} = \frac{\vec{q}}{|\vec{q}|}\,\text{arctanh}\frac{|\vec{q}|}{q_0} \\ q_0 = \sqrt{m^2+\vec{q}^2} \\ s\left(\frac{q}{m}\right) m\mathbf{1}_2 s^*\left(\frac{q}{m}\right) = q_j\sigma^j \end{cases}$$

In the case of polarization **SO**(2), the transition from a space system with the distinguished polarization axis as third direction to a rotation group action compatible framework is performed with the two-sphere representations, parametrized by the two real numbers in the momenta $\frac{\vec{q}}{|\vec{q}|}$

$$u\left(\frac{\vec{q}}{|\vec{q}|}\right) = e^{i\frac{\vec{\alpha}\vec{\sigma}}{2}} = \sqrt{\frac{|\vec{q}|+q^3}{2|\vec{q}|}}\left[\mathbf{1}_2 + i\frac{\vec{q}_\perp\vec{\sigma}}{|\vec{q}|+q^3}\right] \in \mathbf{SU}(2)$$

$$\text{with} \begin{cases} \vec{\alpha} = \frac{\vec{q}_\perp}{|\vec{q}_\perp|}\,\text{arctan}\frac{|\vec{q}_\perp|}{|\vec{q}|} \\ \vec{q}_\perp = (q_2, -q_1, 0) \\ u\left(\frac{\vec{q}}{|\vec{q}|}\right)|\vec{q}|\sigma^3 u^*\left(\frac{\vec{q}}{|\vec{q}|}\right) = \vec{q}\vec{\sigma} \end{cases}$$

Such linear representations of coset representatives, here $s(\frac{q}{m})$ and $\hat{s}(\frac{q}{m}) = s^{-1*}(\frac{q}{m})$ for the boosts $\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3)$ in the two fundamental Weyl representations (often introduced as solutions of the Dirac equation) and $u(\frac{\vec{q}}{|\vec{q}|})$ for the two-sphere $\mathbf{SU}(2)/\mathbf{SO}(2) \cong \mathbf{SO}(3)/\mathbf{SO}(2)$ in the fundamental Pauli representation, will be called transmutators. They have a characteristic hybrid transformation property: The left action with the subgroup gives the transmutator for the transformed momenta up to a right action with the subgroup

$$\lambda \in \mathbf{SL}(\mathbb{C}^2) \Rightarrow \lambda \circ s\left(\frac{q}{m}\right) = s\left(\Lambda(\lambda)\cdot\frac{q}{m}\right)\circ u \quad \text{with} \quad u = u\left(\frac{q}{m},\lambda\right) \in \mathbf{SU}(2)$$

$$v \in \mathbf{SU}(2) \Rightarrow v \circ u\left(\frac{\vec{q}}{|\vec{q}|}\right) = u\left(O(v)\cdot\frac{\vec{q}}{|\vec{q}|}\right)\circ a \quad \text{with} \quad a = a\left(\frac{\vec{q}}{|\vec{q}|}, v\right) \in \mathbf{SO}(2)$$

External transmutators show up in the harmonic (Fourier) analysis of quantum fields with respect to the particle–antiparticle (u, a) creation and annihilation operators involved, for example for the left and right handed Weyl component of a Dirac electron field

$$\mathbf{\Psi}(x) = \begin{pmatrix} \mathbf{r}^{\dot{A}} \\ \mathbf{l}^{A} \end{pmatrix}(x) = \int \frac{d^3q}{(2\pi)^3} \begin{pmatrix} s\left(\frac{q}{m}\right)^{\dot{A}}_{C} \frac{e^{xiq}\mathrm{u}^C(\vec{q}) + e^{-xiq}\mathrm{a}^{\star C}(\vec{q})}{\sqrt{2}} \\ \hat{s}\left(\frac{q}{m}\right)^{A}_{C} \frac{e^{xiq}\mathrm{u}^C(\vec{q}) - e^{-xiq}\mathrm{a}^{\star C}(\vec{q})}{\sqrt{2}} \end{pmatrix}$$

The infinite dimensionality ($\mathbb{R}^3$ cardinality) of the definite unitary representations of the noncompact Poincaré group is seen in the momentum integral $\int \frac{d^3q}{(2\pi)^3} \cong \bigoplus_{\vec{q}\in\mathbb{R}^3}$ over all transmutators.

Higher spin and polarization fields, for example the massive weak vector bosons or the massless electromagnetic vector potential, need transmutators that are products of the two fundamental Weyl transmutators and the fundamental Pauli transmutator respectively, for example

$$\Lambda\left(\frac{q}{m}\right)^{i}_{j} = \frac{1}{2}\,\mathrm{tr}\, s\left(\frac{q}{m}\right)\sigma^i s^*\left(\frac{q}{m}\right)\check{\sigma}_j \quad \in \mathbf{SO}_0(1,3)$$

$$O\left(\frac{\vec{q}}{|\vec{q}|}\right)^{a}_{b} = \frac{1}{2}\,\mathrm{tr}\, u\left(\frac{\vec{q}}{|\vec{q}|}\right)\sigma^a u^*\left(\frac{\vec{q}}{|\vec{q}|}\right)\sigma^b \in \mathbf{SO}(3)$$

## 3. INTERNAL TRANSMUTATION FROM PARTICLES TO INTERACTIONS

In addition to the external rotation and translation properties particles are characterized also by particle–antiparticle $\mathbf{U}(1)$ symmetries, for example the electromagnetic charge number or a fermion–antifermion number, for example for the neutrinos or the neutron. In the standard model of electroweak interactions the electromagnetic real one-dimensional abelian internal $\mathbf{U}(1)$ symmetry is the only remaining symmetry from the real 12-dimensional rank 4 hyperisospin–color group. Particles have no isospin or color symmetry. For example, the proton–neutron doublet displays the isospin multiplicity too, but—with the different masses—no isospin $\mathbf{SU}(2)$ symmetry.

In the standard model, the electromagnetic group $\mathbf{U}(1)$ is the only proper fixgroup ("little group") for the hyperisospin group $\mathbf{U}(2)$ acting on the complex two-dimensional Hilbert space with the Higgs field $\Phi \in \mathbb{C}^2$ with nontrivial scalar product $\|\Phi\|^2 = M^2 > 0$. The internal induction from electromagnetic $\mathbf{U}(1)$ to hyperisospin $\mathbf{U}(2)$

$$\mathbf{rep}\,\mathbf{U}(1) \overset{\mathrm{ind}}{\longmapsto} \mathbf{rep}\,\mathbf{U}(2)$$

is in analogy to the external inductions. The analogy to the rest systems, defined by $q_j\sigma^j = m\mathbf{1}_2$ up to rotations $\mathbf{SO}(3)$, and the polarization systems, defined by $\vec{q}\vec{\sigma} = |\vec{q}|\sigma^3$ up to axial rotations $\mathbf{SO}(2)$, is the electromagnetic system that is defined by $\binom{\Phi_1}{\Phi_2} = \binom{0}{M}$ up to electromagnetic transformations $e^{i(\mathbf{1}_2+\tau^3)\gamma_0} \in \mathbf{U}(1)_+$. The internal induction employs the Higgs-field–defined transformation

$$v\left(\frac{\Phi}{M}\right) = \frac{1}{M}\begin{pmatrix} \Phi^{\star 2} & \Phi_1 \\ -\Phi^{\star 1} & \Phi_2 \end{pmatrix} \in \mathbf{U}(2) \quad \text{with} \quad \begin{cases} \|\Phi\|^2 = M^2 > 0 \\ v\left(\frac{\Phi}{M}\right)\begin{pmatrix} 0 \\ M \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \end{cases}$$

from the electromagnetic system to the hyperisospin $\mathbf{U}(2)$ compatible framework. The Goldstone manifold $\mathbf{U}(2)/\mathbf{U}(1)_+$ involved is parametrized with the three real parameters in $\frac{\Phi}{M}$. The hybrid transformation looks like

$$u \in \mathbf{U}(2) \Rightarrow u \circ v\left(\frac{\Phi}{M}\right) = v\left(u \cdot \frac{\Phi}{M}\right) \circ t \quad \text{with} \quad t = t(u) \in \mathbf{U}(1)_+$$

The transition from the interaction parametrizing fields with hyperisospin symmetry to the particle electromagnetic symmetry is performed with the Higgs transmutator $v(\frac{\Phi}{M})$ (in analogy to the Weyl–Pauli transmutators), for example from the left handed lepton isodoublet $(\mathbf{L}_\alpha)_{\alpha=1,2}$ to the left handed components for the charged massive lepton field and its neutrino which, in turn, are transmuted to their particle systems as described in the last section

$$\mathbf{L}_\alpha^A = v\left(\frac{\Phi}{M}\right)_\alpha \begin{pmatrix} \nu^A \\ \mathbf{l}^A \end{pmatrix}, \quad \begin{cases} \nu^A(x) = \ldots \\ \mathbf{l}^A(x) = \ldots \end{cases}$$

In contrast to the external case only compact groups are involved. Their irreducible representations are finite-dimensional. Therefore there is no analogue to the momentum integral, necessary for the infinite-dimensional representation of the external noncompact groups.

Higher isospin fields, for example the isotriplet gauge field, need transmutators that are products of the fundamental Higgs transmutator, for example

$$O\left(\frac{\Phi}{M}\right)^a_b = \frac{1}{2}\mathrm{tr}\, v\left(\frac{\Phi}{M}\right)\tau^a v^*\left(\frac{\Phi}{M}\right)\tau^b \in \mathbf{SO}(3)$$

## 4. THE OPERATIONAL TRIUNIT: INTERNAL–SPACETIME–EXTERNAL

The transition from the large operational symmetry group of the standard model interactions to the small symmetry groups of the related particles involves the external Weyl–Pauli transmutations and the internal Higgs transmutation

$$\begin{aligned}&\mathbf{rep}\,[\mathbf{U}(1)\times\mathbb{R}\times\mathbf{SU}(2)]\uplus\mathbf{rep}[\mathbf{U}(1)\times\mathbb{R}\times\mathbf{SO}(2)]\\&\overset{\text{ind}}{\longmapsto}\mathbf{rep}[\mathbf{U}(2)\times[\mathbb{R}^4\overset{\leftarrow}{\times}\mathbf{SL}(\mathbb{C}^2)]]\end{aligned}$$

If $\mathbf{SU}(3)$ color fields are included the right hand side has to be written with the hyperisospin-color group (Hucks, 1991; Saller, 1992, 1993, 1994, 1998) whose three factors are correlated via the centrum $\mathbb{I}(2)\times\mathbb{I}(3)=\mathbb{I}(6)=\{z\in\mathbb{Z}\mid z^6=1\}$ of the nonabelian factor

$$\mathbf{rep}\,\mathbf{U}(1)\overset{\text{ind}}{\longmapsto}\mathbf{rep}\,\mathbf{U}(2\times 3)\quad\text{with}\quad\mathbf{U}(2\times 3)=\frac{\mathbf{U}(1)\times\mathbf{SU}(2)\times\mathbf{SU}(3)}{\mathbf{I}(2)\times\mathbf{I}(3)}$$

For the following considerations the color group is excluded. It cannot be described in the structures below, its occurence has to be explained differently, for example as proposed in Saller (1998).[4]

The three factors in the standard model interaction symmetry $\mathbf{U}(2)\times[\mathbb{R}^4\overset{\leftarrow}{\times}\mathbf{SL}(\mathbb{C}^2)]$ describe the internal operations, the spacetime translations and the homogeneous external operations respectively. Such a product constitutes a characteristic structure (Folland, 1995; Fulton and Harris, 1991; Mackey, 1968) occuring for representations of a group $G$ induced by representations of a subgroup $U\subseteq G$. In the representation induction, which will be described in more detail below, the group $G$ is decomposed into disjoint subgroup $U$ orbits and representatives $(U\backslash G)_{\text{repr}}$ for the cosets $U\backslash G$

$$G=U\times(U\backslash G)_{\text{repr}}=\biguplus_{\text{repr}\,k_r\in G}Uk_r$$

For notational convenience the left classes $Uk$, that is the $U$ orbits under left multiplication are taken

$$u\in U:L_u:G\longrightarrow G,\qquad L_u(k)=uk$$

To establish the standard model operations as an example for the abstract structure

$$\mathbf{U}(2)\times[\mathbb{R}^4\overset{\leftarrow}{\times}\mathbf{SL}(\mathbb{C}^2)]\overset{?}{\sim}U\times[(U\backslash G)_{\text{repr}}\overset{\leftarrow}{\times}G]$$

the Lorentz group cover $\mathbf{SL}(\mathbb{C}^2)$ is filled up by a phase $\mathbf{U}(1)$ group (fermion number) and a dilatation group $\mathbf{D}(1)$ (causal group) to the full linear group $\mathbf{GL}(\mathbb{C}^2)$, a real eight-dimensional Lie group

$$\mathbf{GL}(\mathbb{C}^2)=\mathbf{D}(\mathbf{1}_2)\times\mathbf{UL}(2)$$

[4] Also see H. saller, hep-th/0010057.

The direct unimodular factor involved is the centrally correlated product of two normal subgroups, the fermion number and the Lorentz covering group

$$\mathbf{UL}(2) = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SL}(\mathbb{C}^2) = \{g \in \mathbf{GL}(\mathbb{C}^2) \mid |\det g| = 1\}$$

$$\mathbf{U}(\mathbf{1}_2) \cap \mathbf{SL}(\mathbb{C}^2) = \mathbb{I}(2) = \{\pm \mathbf{1}_2\}, \quad \begin{cases} \mathbf{UL}(2)/\mathbf{SL}(\mathbb{C}^2) \cong \mathbf{U}(1) \\ \mathbf{UL}(2)/\mathbf{U}(\mathbf{1}_2) \cong \mathbf{SL}(\mathbb{C}^2)/\mathbb{I}(2) \\ \qquad\qquad\qquad \cong \mathbf{SO}_0(1,3) \end{cases}$$

Therewith the triad $U \times [(U\backslash G)_{\mathrm{repr}} \overleftarrow{\times} G]$ of the internal–spacetime–external transformations will be defined with a maximal compact subgroup $\mathbf{U}(2)$, defining the internal operations, in the full group $\mathbf{GL}(\mathbb{C}^2)$, defining the external operations

$$\text{operational triunit: } \mathbf{U}(2) \times [\mathbf{D}(2) \overleftarrow{\times} \mathbf{GL}(\mathbb{C}^2)]$$

The manifold of hyperisospin $\mathbf{U}(2)$ orbits in the full external group $\mathbf{GL}(\mathbb{C}^2)$ is a real four-dimensional rank 2 symmetric space $\mathbf{D}(2)$, which will be used as a model for nonlinear spacetime (Saller, 1997, 1998b). It has as representatives the hermitian invertible $2 \times 2$ matrices, which can also be parametrized by the translations of the strictly future lightcone

$$(\mathbf{U}(2)\backslash\mathbf{GL}(\mathbb{C}^2))_{\mathrm{repr}} = \mathbf{D}(2) = \left\{ k \in \mathbf{GL}(\mathbb{C}^2) \mid k = k^* = \begin{pmatrix} k_0 + k_3 & k_1 - ik_2 \\ k_1 + ik_2 & k_0 - k_3 \end{pmatrix} \right.$$
$$\left. \text{and} \quad \mathrm{spec}\, k > 0 \right\}$$

All $2 \times 2$ matrices with $\mathbf{U}(2)$ conjugation constitute a $C^*$ algebra with the natural spectral order and the polar decomposition of the full group into internal compact operations and noncompact spacetime

$$\mathbf{GL}(\mathbb{C}^2) = \mathbf{U}(2) \times \mathbf{D}(2), \quad k = u \circ |k|, \quad |k| = \sqrt{k^* \circ k}$$

In the general structure, the group $G$ acts on the left orbits $Uk$ of a subgroup $U$ by right inverse multiplication, which may look quite complicated for the chosen orbit representatives

$$g \in G : R_g : U\backslash G \longrightarrow U\backslash G, \quad R_g(Uk) = Ukg^{-1}$$
$$(U\backslash G)_{\mathrm{repr}} \longrightarrow (U\backslash G)_{\mathrm{repr}}, \quad k_r \longmapsto k_{r'} \quad \text{for} \quad k_r g^{-1} = uk_{r'}$$
$$\text{with} \quad u = u(k_r, g^{-1}) \in U$$

In the physical structure proposed one obtains the action of the full external group $\mathbf{GL}(\mathbb{C}^2)$ on the nonlinear spacetime $\mathbf{D}(2)$

$$g \in \mathbf{GL}(\mathbb{C}^2) : \mathbf{D}(2) \longrightarrow \mathbf{D}(2), \quad |k| \longmapsto |k'| \quad \text{for} \quad |k| \circ g^{-1} = u \circ |k'|$$
$$\text{with} \quad u = u(|k|, g^{-1}) \in \mathbf{U}(2) \Rightarrow |k'| = \sqrt{g^{-1*} \circ |k|^2 \circ g^{-1}} = |k \circ g^{-1}|$$

The tangent space of the symmetric space $\mathbf{D}(2)$ constitutes the spacetime translations with the faithful action of the causal Lorentz group

$$\begin{aligned}\log \mathbf{D}(2) &= \{x = x_j\sigma^j \mid e^x = |k| \in \mathbf{D}(2)\} \cong \mathbb{R}^4\\ g &= e^{\frac{\beta_0+i\alpha_0}{2}} s \in \mathbf{GL}(\mathbb{C}^2) : x \longmapsto g \circ x \circ g^*\\ &\Rightarrow \frac{1}{2}\mathrm{tr}\; g\sigma^i g^* \check{\sigma}_j = e^{\beta_0}\Lambda(s) \in \mathbf{D}(1) \times \mathbf{SO}_0(1,3) \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(\mathbf{1}_2)\end{aligned}$$

## 5. INTERNAL–EXTERNAL ACTIONS ON STANDARD MODEL FIELDS

The transformation behavior of fields with respect to external Lorentz and internal hyperisospin operations is quite differents: The fields used in the standard model with the operations $\mathbf{U}(2) \times [\mathbb{R}^4 \overset{\leftarrow}{\times} \mathbf{GL}(\mathbb{C}^2)]$, for example the left handed lepton isodoublet $\{\mathbf{L}^A_\alpha\}^{A=1,2}_{\alpha=1,2}$, map the spacetime translations $\mathbb{R}^4$ into a complex vector space

$$\mathbf{L}^A_\alpha : \mathbb{R}^4 \longrightarrow W \otimes V^T, \qquad x \longmapsto \mathbf{L}^A_\alpha(x)$$

The value space is the tensor product of a finite-dimensional space $W$ with the representation of hyperisospin $\mathbf{U}(2)$, in the lepton isodoublet example the defining representation on $W \cong \mathbb{C}^2$ with $\mathbf{U}(1)$-hypercharge number $y = -1/2$

$$\begin{aligned}D : \mathbf{U}(2) &\longrightarrow \mathbf{GL}(W), &\qquad D(u) &= u = e^{\frac{-i\gamma_0\mathbf{1}_2+i\vec{\gamma}\vec{\tau}}{2}}\\ \mathbf{U}(2) \times W &\longrightarrow W, & u \cdot \mathbf{L}^A_\alpha &= u^\beta_\alpha \mathbf{L}^A_\beta\end{aligned}$$

and another finite-dimensional vector space $V$ with a Lorentz group representation, in the example the defining left handed Weyl representation on $V \cong \mathbb{C}^2$

$$T : \mathbf{SL}(\mathbb{C}^2) \longrightarrow \mathbf{GL}(V), \qquad T(s) = s = e^{(i\vec{\alpha}+\vec{\beta})\frac{\vec{\sigma}}{2}}$$

Since the spacetime translations $\mathbb{R}^4$ are also acted upon with the Lorentz group, the field as a mapping between two vector spaces with Lorentz group action transforms $\mathbf{L} \longmapsto \mathbf{L}_s$ as given by the commutative diagram Bourbaki (1989)

$$\begin{array}{ccc} \mathbb{R}^4 & \xrightarrow{\Lambda(s)} & \mathbb{R}^4\\ \mathbf{L}\downarrow & & \downarrow \mathbf{L}_s\\ V^T & \xrightarrow[s]{} & V^T \end{array}, \qquad \begin{aligned}\Lambda &= \Lambda(s) \in \mathbf{SO}_0(1,3)\\ \mathbf{L}_s(\Lambda \cdot x) &= s \cdot \mathbf{L}(x)\end{aligned}$$

$$\mathbf{SL}(\mathbb{C}^2) \times V^T \longrightarrow V^T, \qquad (\mathbf{L}_s)^A_\alpha(x) = \mathbf{L}^B_\alpha(\Lambda^{-1} \cdot x) s^A_B$$

For notational convenience the dual space $V^T$ (linear $V$ forms) is used.

Both internal and external transformation behavior can be collected into one diagram, for example for the lepton isodoublet left handed Weyl field here

$$\begin{array}{ccccc} & \mathbb{R}^4 & \xrightarrow{\boldsymbol{\Lambda}(s)} & \mathbb{R}^4 & \\ \mathbf{L} & \downarrow & & \downarrow & \mathbf{L}_s\,, \\ & W \otimes V^T & \xrightarrow[u\otimes s]{} & W \otimes V^T & \end{array} \qquad (\mathbf{L}_s)^A_\alpha(x) = u^\beta_\alpha \mathbf{L}^B_\beta(\Lambda^{-1}\cdot x)s^A_B$$

or for the isotriplet gauge vector field $\{\mathbf{A}^j_a\}^{j=0,1,2,3}_{a=1,2,3}$ valued in the vector space $W' \otimes V'^T \cong \mathbb{C}^3 \otimes \mathbb{C}^4$

$$\begin{array}{ccccc} & \mathbb{R}^4 & \xrightarrow{\boldsymbol{\Lambda}(s)} & \mathbb{R}^4 & \\ \mathbf{A} & \downarrow & & \downarrow & \mathbf{A}_s\,, \\ & W' \otimes V'^T & \xrightarrow[\boldsymbol{O}(u)\otimes\Lambda(s)]{} & W' \otimes V'^T & \end{array} \qquad \begin{array}{c} (\mathbf{A}_s)^j_a(x) = O^b_a \mathbf{A}^k_b(\Lambda^{-1}\cdot x)\Lambda^j_k \\ O = O(u) \in \mathbf{SO}(3) \end{array}$$

These transformation properties are compared in the next sections with the transformation properties occuring for induced representations.

## 6. INDUCED REPRESENTATIONS

The structure of induced representations as used for example for Wigner's particles classification can be sketched for our purposes—without discussion of topological structures—as follows (Folland, 1995; Fulton and Harris, 1991; Mackey, 1968):

A group $G$ representation induced by the representation of a subgroup $D: U \longrightarrow \mathbf{GL}(W)$ on a complex vector space acts on the subgroup interwiners, that is on the mappings from the group $G$ into the vector space $W$, compatible with the action of $U$ on $G$ by left multiplication and on $V$ by the representation $D$

$$\begin{array}{ccccc} & G & \xrightarrow{L_u} & G & \\ w & \downarrow & & \downarrow & w\,, \\ & W & \xrightarrow[D(u)]{} & W & \end{array} \qquad w(uk) = D(u)\cdot w(k) \quad \text{for all } u \in U,\ k \in G$$

The intertwiner space dimensionality is the product of the $W$ dimensionality with the cardinality of the $U$ orbits, that is in general infinite for Lie groups

$$\dim_{\mathbb{C}} W_U(G) = \dim_{\mathbb{C}} W \cdot \mathrm{card}\, U\backslash G$$

The group $G$ action on the vector space with the intertwiners $w \in W_U(G)$ is defined by the following commutative diagram, which involves the right inverse multiplication $k \longmapsto kg^{-1}$ on the group $G$, not used in the definition of the intertwiners

$$\begin{array}{ccc} G & \stackrel{R_g}{\longrightarrow} & G \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle w_g}, \\ W & \underset{\mathrm{id}_W}{\longrightarrow} & W \end{array} \qquad w_g(k) = w(kg) \quad \text{for all } k \in G,\ g \in G$$

$$G \times W_U(G) \longrightarrow W_U(G), \qquad w \longmapsto w_g$$

Again, both diagrams can be taken together. With a decomposition into $U$ orbits and representatives $G = U \times (U\backslash G)_{\mathrm{repr}} = \biguplus_r Uk_r$ the induced $G$ representation reads

$$\begin{array}{ccc} G & \stackrel{L_u \circ R_g}{\longrightarrow} & G \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle w_g}, \\ W & \underset{D(u)\circ \mathrm{id}_W}{\longrightarrow} & W \end{array} \qquad \begin{array}{rcll} w_g(uk) & = & D(u)\cdot w(kg) & \text{for all } u \in U,\ k \in G, g \in G \\ w_g(k_r) & = & D(u)\cdot w(k_{r'}) & \text{for} \quad k_r g = uk_{r'} \quad \text{with} \\ & & & \quad u = u(k_r, g) \in U \end{array}$$

## 7. TRANSMUTATORS

In general, an induced $G$ representation is infinite-dimensional and—in many cases, for example for compact groups—highly reducible, for example the right regular representation on the algebra $\mathbb{C}(G) = \{G \longrightarrow \mathbb{C}\}$ with the group functions, which is induced by the trivial representation of the trivial subgroup $U = \{e\}$ on the numbers $\mathbb{C}$, or the $G$ representation on an intertwiner space $W_U(G)$.

The group functions $\mathbb{C}(G)$ contain—up to isomorphy—the representation space of each finite-dimensional $G$ representation

$$T\ :\ G \longrightarrow \mathbf{GL}(V)$$

via the representation matrix elements, isomorphic to $V \otimes V^T$

$$T(g)\ :\ V \longrightarrow V$$

$$V \otimes V^T \cong \left\{T^v_\omega \;\middle|\; v \in V, \omega \in V^T\right\} \subset \mathbb{C}(G) \quad \text{with} \quad \begin{cases} T^v_\omega : G \longrightarrow \mathbb{C} \\ T^v_\omega(k) = \langle \omega, T(k)\cdot v\rangle \\ T^v_\omega(kg) = T^{T(g)\cdot v}_\omega(k) \end{cases}$$

A decomposition of a $G$ representation into $U$ representations with projectors $\{\mathcal{P}_\iota\}_\iota$

$$V = \bigoplus_\iota W_\iota, \quad T[U]\cdot W_\iota \subseteq W_\iota, \quad T|_U = D = \bigoplus_\iota D_\iota$$

$$\mathcal{P}_\iota : V \longrightarrow W_\iota, \quad D_\iota\ :\ U \longrightarrow \mathbf{GL}(W_\iota), \quad D_\iota(u)\ :\ W_\iota \longrightarrow W_\iota$$

and an orbit decomposition of the full group $G = U \times (U\backslash G)_{\mathrm{repr}} = \biguplus_r Uk_r$ give rise to transmutators that are valued in the tensors $W_\iota \otimes V^T$ as products of the $G$ space $V$ and a $U$ subspace $W_\iota$

$$T_\iota : G \longrightarrow W_\iota \otimes V^T, \qquad T_\iota(uk_r) = D_\iota(u) \circ \mathcal{P}_j \circ T(k_r) : V \longrightarrow W_\iota$$

If $V \cong \mathbb{C}^n$, then $T(k)$ has an $n \times n$ matrix form. If $W_\iota \cong \mathbb{C}^m$ with $m \leq n$, then $D_\iota(u)$ has an $m \times m$ matrix form and $T_\iota(k_r)$ an $m \times n$ matrix form.

All "right-sided" matrix elements of a transmutator constitute a $G$-stable subspace of the $U$ intertwiners

$$W_\iota \otimes V^T \cong \left\{T_\iota^v \,\middle|\, v \in V\right\} \subset W_{\iota,U}(G) \quad \text{with} \quad \begin{cases} T_\iota^v : G \longrightarrow W_\iota \\ T_\iota^v(uk_r) = D_\iota(u) \circ \mathcal{P}_j \circ T(k_r) \cdot v \\ T_\iota^v(kg) = T_\iota^{T(g)\cdot v}(k) \end{cases}$$

Therewith the intertwiner space $W_U(G)$ contains—up to isomorphy—all tensor products $W \otimes V^T$, where $V$ is acted on with a finite-dimensional subrepresentation of the full group $G$

$$D[U] \cdot W \subseteq T[G] \cdot V \Rightarrow W \otimes V^T \hookrightarrow W_U(G)$$

$U\backslash G$ transmutators for irreducible $G$ representations are building blocks of induced representations. They transform from a vector space $V$ with the action of a group $G$ to a vector subspace $W$ with the action of a subgroup $U$. Transmutators with $W = V$ are called complete, that is all $U$ representations contained in the $G$ representation are included. Complete transmutators are bijections.

## 8. FIELDS AS INTERNAL–EXTERNAL TRANSMUTATORS

Spacetime fields $\boldsymbol{\Psi}$ for the operational triunit $U \times [(U\backslash G)_{\mathrm{repr}} \overset{\leftarrow}{\times} G]$ will be defined to be transmutators from external group $G$ representations on a vector space $V$ to internal subgroup $U$ representations on a vector subspace $W$. They are parametrized with the orbit manifold $U\backslash G$ of the possible $U$'s in $G$

$$\boldsymbol{\Psi} : (U\backslash G)_{\mathrm{repr}} \longrightarrow W \otimes V^T, \quad \begin{cases} U \times W \longrightarrow W \ \text{(internal)} \\ G \times V \longrightarrow V \ \text{(external)} \\ U \subseteq G, \quad W \subseteq V \end{cases}$$

The geometrical structure can be formulated also in a bundle language.

The internal hyperisospin group $\mathbf{U}(2)$ is a maximal compact subgroup of the external group $\mathbf{GL}(\mathbb{C}^2) = \mathbf{D}(\mathbf{1}_2) \times \mathbf{UL}(2)$ with the causal group and the unimodular fermion-number–Lorentz-group cover $\mathbf{UL}(2) = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SL}(\mathbb{C}^2)$ as direct factors. Nonlinear spacetime $\mathbf{D}(2)$ parametrizes the noncompact manifold $\mathbf{U}(2)\backslash\mathbf{GL}(\mathbb{C}^2)$.

### 8.1. The Fundamental Transmutator on Nonlinear Spacetime

The fundamental spacetime field for the operational triunit

$$\mathbf{U}(2) \times [\mathbf{D}(2) \overleftarrow{\times} \mathbf{GL}(\mathbb{C}^2)]$$

transmutes from the defining internal $\mathbf{U}(2)$-isodoublet space $W \cong \mathbb{C}^2$ to the defining external $\mathbf{SL}(\mathbb{C}^2)$-Weyl spinor space $V \cong \mathbb{C}^2$

$$\Psi^A_\alpha : \mathbf{D}(2) \longrightarrow W \otimes V^T, \quad |k| \longmapsto \Psi^A_\alpha(|k|)$$

It has the internal $\mathbf{U}(2)$ and the external $\mathbf{GL}(\mathbb{C}^2)$ transformation behavior

$$\mathbf{U}(2) \times W \longrightarrow W, \quad \Psi^A_\alpha \longmapsto u^\beta_\alpha \Psi^A_\beta$$
$$\mathbf{GL}(\mathbb{C}^2) \times V \longrightarrow V, \quad \Psi^A_\alpha(|k|g) = \Psi^B_\alpha(|k|)g^A_B = u(|k|, g)^\beta_\alpha \Psi^A_\beta(|k \circ g|)$$

Since the nonlinear spacetime manifold can be parametrized as the strictly future lightcone $\mathbf{D}(2) \cong \mathbb{R}^4_\succ \subset \mathbb{R}^4$, $|k| = x_\succ$, of its tangent space, the spacetime translations $\log \mathbf{D}(2) \cong \mathbb{R}^4$, the fundamental isospinor Weyl spinor field has causal support without spacelike particle interpretable contributions. Its spectrum with respect to the action of the causal group $\mathbf{D}(1)$ has to be investigated to find its particle interpretable content that can be defined for all spacetime translations $\mathbb{R}^4$. First steps on this way have been tried previously.[5]

The fundamental isospinor–spinor dyad $\{\Psi^A_\alpha\}^{A=1,2}_{\alpha=1,2}$ for the hyperisospin $\mathbf{U}(2)$ orbits in the extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$ can be seen in some analogy (saller, 1998b) to the tetrad $\{\mathbf{h}^\mu_j\}^{\mu=0,1,2,3}_{j=0,1,2,3}$ in general relativity for the orbits of the Lorentz group $\mathbf{SO}_0(1,3)$ in the general linear group $\mathbf{GL}(\mathbb{R}^4)$.

### 8.2. Standard Model Fields as Transmutators on Linear Spacetime

Without being able so far to determine the spectrum of the causal group action on the fundamental transmutator for a particle interpretation one may start less ambitiously and try to interpret the standard model fields as a linear approximation, that is as internal–external transmutators parametrized with spacetime translations $\log \mathbf{D}(2) \cong \mathbb{R}^4$

$$\mathbf{U}(2) \times [\mathbb{R}^4 \overleftarrow{\times} \mathbf{GL}(\mathbb{C}^2)]$$

Any representation of a group $D : G \longrightarrow \mathbf{GL}(V)$ is faithful up to its kernel, a normal $G$ subgroup, that is $D[G] \cong G/\text{kern } D$. Therefore the representations of the internal hyperisospin group $\mathbf{U}(2) = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SU}(2)$ with $\mathbf{U}(\mathbf{1}_2) \cap \mathbf{SU}(2) = \mathbb{I}(2)$ have as nontrivial images three groups – the full hyperisospin, the hypercharge,

[5] H. saller, hep-th/0010057.

and the isorotation group

$$\mathbf{U}(2)\text{-representation images: } \mathbf{U}(2),\ \mathbf{U}(1) \cong \mathbf{U}(2)/\mathbf{SU}(2)$$
$$\mathbf{SO}(3) \cong \mathbf{U}(2)/\mathbf{U}(\mathbf{1}_2)$$

to be compared with the three nontrivial representation images of the external unimodular group, given by the full group, the fermion number and the Lorentz group

$$\mathbf{UL}(2)\text{-representation images: } \mathbf{UL}(2),\ \mathbf{U}(1) \cong \mathbf{UL}(2)/\mathbf{SL}(\mathbb{C}^2)$$
$$\mathbf{SO}_0(1,3) \cong \mathbf{UL}(2)/\mathbf{U}(\mathbf{1}_2)$$

There are three nontrivial internal–external embeddings – hyperisospin $\mathbf{U}(2)$ and hypercharge $\mathbf{U}(1)$ into the fermion-number–Lorentz-group $\mathbf{UL}(2)$ and isorotations $\mathbf{SO}(3)$ into the Lorentz group $\mathbf{SO}_0(1, 3)$

$$\mathbf{U}(2) \hookrightarrow \mathbf{UL}(2), \quad \mathbf{U}(1) \hookrightarrow \mathbf{UL}(2), \quad \mathbf{SO}(3) \hookrightarrow \mathbf{SO}_0(1,3)$$

In the standard model the left handed Weyl isodoublet field $\mathbf{L}$, the right handed Weyl isosinglet fields $\mathbf{R}$ and the Lorentz vector isosinglet–isotriplet gauge fields $\mathbf{A}$ are the corresponding transmutators, as mappings from the coset tangent space $\log(U\backslash G)_{\text{rep}} \longrightarrow W \otimes V^T$ into an internal–external vector space tensor product with the faithful action of the represented images $D[U] \otimes T[G]$

$$\begin{aligned}
\mathbf{L} : \mathbb{R}^4 &\longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \quad \text{with} \quad \mathbf{U}(2) \otimes \mathbf{UL}(\mathbb{C}^2)\\
x &\longmapsto \mathbf{L}^A_\alpha(x), \qquad \alpha = 1, 2;\ A = 1, 2\\
\mathbf{R} : \mathbb{R}^4 &\longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \quad \text{with} \quad \mathbf{U}(1) \otimes \mathbf{UL}(\mathbb{C}^2)\\
x &\longmapsto \mathbf{R}^{\dot A}_{1,2}(x), \qquad \dot A = 1, 2\\
\mathbf{A} : \mathbb{R}^4 &\longrightarrow \mathbb{C}^4 \otimes \mathbb{C}^4 \quad \text{with} \quad \mathbf{SO}(3) \times \mathbf{SO}_0(1,3)\\
x &\longmapsto \mathbf{A}^j_{0,a}(x), \qquad a = 1, 2, 3;\ j = 0, 1, 2, 3
\end{aligned}$$

There are two fermionic and one bosonic transmutator. With coinciding internal and external representation space all three transmutators are complete. The right handed two-component Weyl field $\mathbf{R}$ comprises two isosinglets $\{\mathbf{R}_1, \mathbf{R}_2\}$, and the four-component Lorentz vector field $\mathbf{A}$ four internal degrees of freedom, an isosinglet and an isotriplet $\{\mathbf{A}_0, \vec{\mathbf{A}}\}$.

The transition from those standard fields for the interactions to particles for the state space requires internal transmutators, parametrized with the Higgs degrees of freedom (Goldstone manifold), as discussed previously herein,

$$\begin{aligned}
v : (\mathbf{U}(2)/\mathbf{U}(1)_+)_{\text{repr}} &\longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \quad \text{with} \quad \mathbf{U}(1) \otimes \mathbf{U}(2)\\
\frac{\Phi}{M} &\longmapsto v\left(\frac{\Phi}{M}\right)^{1,2}_{\alpha}, \qquad \alpha = 1, 2
\end{aligned}$$

and external Weyl–Pauli transmutators, parametrized with the momenta as coset representatives (boost manifold, two-sphere)

$$s, \hat{s} \; : \; (\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2))_{\rm repr} \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \quad \text{with} \quad \mathbf{SU}(2) \otimes \mathbf{SL}(\mathbb{C}^2)$$

$$\frac{q}{m} \longmapsto s\left(\frac{q}{m}\right)_C^{\dot{A}}, \; \hat{s}\left(\frac{q}{m}\right)_C^{A}, \qquad C = 1, 2; \; \dot{A}, A = 1, 2$$

$$u \; : \; (\mathbf{SU}(2)/\mathbf{SO}(2))_{\rm repr} \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \quad \text{with} \quad \mathbf{SO}(2) \otimes \mathbf{SU}(2)$$

$$\frac{\vec{q}}{|\vec{q}|} \longmapsto u\left(\frac{\vec{q}}{|\vec{q}|}\right)_{1,2}^{\alpha}, \qquad \alpha = 1, 2$$

The operational triunits for the internal and external interaction-particle transmutations are

$$\text{Higgs:} \quad \mathbf{U}(1) \times [(\mathbf{U}(2)/\mathbf{U}(1)_+)_{\rm repr} \overleftarrow{\times} \mathbf{U}(2)]$$

$$\text{Weyl:} \quad \mathbf{SU}(2) \times [(\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2))_{\rm repr} \overleftarrow{\times} \mathbf{SL}(\mathbb{C}^2)]$$

$$\text{Pauli:} \quad \mathbf{SO}(2) \times [(\mathbf{SU}(2)/\mathbf{SO}(2))_{\rm repr} \overleftarrow{\times} \mathbf{SU}(2)]$$

## REFERENCES

Bourbaki, N. (1989). *Lie Groups and Lie Algebras, Chap. 1–3*, Springer, Berlin.
Folland, G. B. (1995). *A Course in Abstract Harmonic Analysis*, CRC Press.
Fulton, W. and Harris, J. (1991). *Representation Theory*, Springer.
Hucks, J. (1991). *Physical Review D* **43**, 2709.
Mackey, G. W. (1968). *Induced Representations of Group and Qauntum Mechanics*, W. A. Benjamin, New York.
Saller, H. (1992). *Nuovo Cimento* **105A**, 1745.
Saller, H. (1993). *Nuovo Cimento* **106A**, 1189.
Saller, H. (1994). *Nuovo Cimento*, **107A**, 1679.
Saller, H. (1998a). *Nuovo Cimento*, **111A**, 1375.
Saller, H. (1997). *International Journal of Theoretical Physics* **36**, 2783.
Saller, H. (1999). *International Journal of Theoretical Physics* **38**, 1697.
Saller, H. (1998b). *International Journal of Theoretical Physics* **37**, 2333.
Wigner, E. P. (1939). *Annals of Mathematics* **40**, 149.